3.7 Transforming Celestial and Terrestrial Coordinates

Orbit determination requires both celestial reference frames (which define the Newtonian-inertial space in which differential equations of satellite motion are valid), and terrestrial reference frames (from which satellite observations are taken). The conventional terrestrial frame is the ITRF, and this is the frame for most observations. The WGS-84 terrestrial frame is used primarily by the US DoD. It’s realized through GPS observations, although the fundamental WGS-84 stations are usually constrained by their adopted ITRF coordinates during solution. Thus, the modern WGS-84 and ITRF terrestrial frames agree at the cm level, so within the uncertainty of the WGS-84 frame, they are practically identical.

The terrestrial frame is related to the celestial frame through a series of translations and rotations known as an Earth orientation model or reduction formulas (Seidelmann, 1992, 20). The complete model is sometimes divided into partial series of rotations, where the intermediate frames are defined between each sequence. There are two conventional Earth orientation models: the classical transformation and the Celestial Ephemeris Origin (CEO). The latter theory is known as the new IAU 2000 Theory. At the time of this writing, the IAU 2000 Theory was just officially released (McCarthy, 2003). This document outlines resolutions which state that beginning January 1, 2003, the IAU 1976 Precession Model and the IAU 1980 Theory of Nutation are replaced by the IAU 2000a theory (accurate to 0.0002°) and the IAU 2000b theory (accurate to 0.001°). Before this time, the recommended reduction to GCRF coordinates remains the IAU 1980 Theory of nutation, plus the corrections (δΔΨ, δΔε) given in the EOP.

This section introduces concepts necessary for the classical transformation and although they don’t directly apply to the IAU 2000, they are beneficial for a general understanding of the topic. The classical transformation exists in numerous computer programs, so it’s also necessary to explain its formation. I won’t discuss translations (topocentric to geocentric to barycentric) because they are simply vector operations. Also see Sec. 4.4.4 for the complete sequence of steps to reduce observations. I separate polar motion and the fundamental arguments because they are common to both transformations.

A truly inertial coordinate frame is difficult to realize. In developing his law of gravitation, Newton described this inertial reference frame as fixed in “absolute space, which in its own nature, without relation to anything external, remains always similar and immovable” (Newton [1687] 1962, 6). He realized that inertial frames can move with constant velocity, but he didn’t show how to find such a frame. This turned out to be a rather difficult problem. In fact, only the extra-galactic coordinate system (origin at the center of the galaxy) approaches Newton’s definition. The solution relies on finding
pseudoinertial reference frames, which we’ll define as a specific system for which
Newtonian mechanics hold. Sec. 3.3 discusses the existence of many different coordinate
systems, some of which may be considered pseudoinertial. Depending on the problem,
we can consider most of the coordinate systems in Sec. 3.3 as pseudoinertial
because the small noninertial effects (1) can’t be measured with current technology, or
(2) are assumed to be negligible during the period of interest. Proper understanding of
these pseudoinertial coordinate systems is needed for highly accurate simulations.

In a stricter sense, determining truly inertial systems is different when we account for
relativity. Solutions in this area use quasi-inertial systems, which have special relativistic forms. Don’t confuse these systems with the pseudoinertial systems sufficient for
Newtonian mechanics. Seidelmann (1992, 96) gives insight concerning quasi-inertial systems:

> There are finite regions with respect to a chosen space of reference where material particles
> move freely without acceleration and in which the laws of special relativity hold with
> remarkable accuracy, which can be regarded as quasi-inertial frames. Thus the center of
> mass of the Earth-Moon system falling in an elliptic orbit around the Sun in a relatively
> weak gravitational field is an example of such a finite region. Therefore, we may assume
> quasi-inertial frames of reference, and any violation of the principles when using classical
> mechanics can be taken into account with small corrections applied to the observations and
> by an appropriate coordinate-time reference. The effects of special relativity for a system
> moving with the Earth around the Sun are of the order $10^{-8}$; likewise those with general rel-
> ativity are of order $10^{-8}$. It is necessary to distinguish between various types of quasi-iner-
> tial frames.

For precise applications, most analyses use a coordinate system which is sufficiently
inertial for the particular application. These systems are ones in which Newton’s laws of
motion are valid. In the case of orbits around the Sun—planets, asteroids, comets and
some deep space probes—the heliocentric coordinate system is usually sufficiently inert-
ial. For satellites orbiting the Earth, the GCRF is sufficient.

The classical transformation relies on the equator and the equinox, but they do
exhibit a small motion, but it is quite small and can be ignored over a few days, depend-
ing on your application. This is not true for longer intervals. If observations from suffi-
ciently different epochs are to be combined for processing, we need a way to transform
between the respective frames, so the observations can be brought to a common epoch.
These different inertial frames are designated by the epoch or date of the equator and
equinox for that frame. To do so, we must develop expressions for the effects of preces-
sion and nutation and designate a standard inertial frame from which all others may be
referred. We can perform these reductions on the current inertial system (GCRF), or its
predecessors, FK5 (J2000), FK4 (B1950), or any other systems of choice.*

To understand the motion of a coordinate system, we begin with the geometry of the
Earth’s orbit. The equatorial planes of the Earth and the ecliptic are not actually fixed

*The FK4 system has noticeable errors, so we should avoid it whenever possible. See Seidelmann
(1992, 167–187) and Moritz and Mueller (1987, 525–567) for other transformations. I’ll talk
briefly about the FK4 system in the upcoming section on FK4 reduction.
relative to the stars. Mathematically, these discrepancies have been divided into a pre-
cessional and a nutational part. First, the gravitational forces of the planets affect the
Earth’s orbit through planetary precession, which results in a very slow secular change
in the ecliptic’s orientation. This causes the equinox to precess to the west about
0.0033°(12”) per century and decreases the obliquity of the ecliptic about
0.013 05°(47”) per century. Next, because the Earth isn’t spherical (dominated by a sig-
nificant equatorial bulge), the Sun and Moon’s gravitational fields produce a very small
torque on the Earth causing luni-solar precession. This torque results in a smooth wob-
bling or precessional motion over a very long period—similar to a spinning top.
Because the obliquity of the ecliptic is about 23.5°, you can see from Fig. 3-23 that the
Earth’s axis of rotation traces out a roughly circular shape over each period of preces-
sion. The half-cone angle equals about 23.5°. The period of the luni-solar precession is
about 26,000 years, and the angle precesses about 0.013 846°(49.846”) per year. The
combined effects of planetary and luni-solar precession are collectively called general
precession, which amounts to about 0.013 889° (50”) per year in longitude.

The Moon produces an additional torque on the Earth’s equatorial bulge causing
nutation—small oscillations in the Earth’s rotation axis. This is because the Moon’s
orbit about the Earth is inclined about 5°, which causes monthly variations in the peri-
odic torque the Moon exerts on the Earth’s equatorial bulge. More importantly, the
Moon’s orbital plane itself precesses with a period of about 18.6 years due to solar per-
trurbations, so nutation from the Moon has a primary period of about 18.6 years. This is
more commonly called the regression of the [lunar] node because the motion is in a
negative direction. Another contribution is the advance of perigee of the lunar orbit due
to solar perturbations. The eccentricity of the Moon’s orbit causes the direction of max-
imum torque to vary. Because the overall nutation effect is actually a complicated result
of more than one disturbance, it has many smaller periods, is periodic, and appears as a
superimposed motion with a maximum amplitude of about 0.0025°(9”) in the obliquity
of the ecliptic, and about 0.004 72°(17”) in longitude. Figure 3-23 shows the result.

The motion of the ecliptic plane due to general precession causes the equinox to
move along [in the plane of] the equator, and we associate its orientation with an ideal
inertial frame on a specific date, the ecliptic of date. The Earth’s rotation axis also pre-
cesses secularly about the North ecliptic pole causing the equator to wobble about the
ecliptic. We model this effect as a mean equator of date measuring the nearly secular
motion of the Earth’s rotation axis. The intersection of these two planes on a given date
defines the mean equinox of date. By including the nutation (periodic small oscillation)
effects of the luni-solar perturbations on the equator, we get the true equator of date.
For precise applications, we must specify the reference frame for all measurements
(Green, 1988, 54). We can also specify each of these frames as of epoch where the
epoch is an external date that fixes the orientation of the coordinate system.

Polar Motion

The Celestial Ephemeris Pole (CEP) is the axis of Earth rotation, which is normal to the
true equator. Seidelmann (1982) defines polar motion as the “movement of the rotation
axis with respect to the crust of the Earth”. The axes of the ITRF define one set of spher-
ical coordinates, and the CEP describes a slightly different set for the same location on the Earth. Because Earth orientation observations are sensitive to the CEP, and not the ITRF (the ITRF is defined by convention, not observation), the recovered latitude and longitude are with respect to the CEP. Unfortunately, this is not the Earth-fixed system we would desire for performing accurate surveys or navigation. The change of basis between these systems is polar motion, and it uses the angles given in the EOP values. The angles (which are part of the Earth Orientation Parameters, EOP) are measured by the International Polar Motion Service (IPMS) and published by International Earth Rotation Service (IERS*), as well as in the Astronomical Almanac (1994, K10, B60). Appendix D lists other sources including the National Geospatial-Intelligence Agency (NGA) predictions, and Table 3-5 shows some recent values. The actual displacements are given in arcseconds in the IERS Reference Meridian (IRM) direction ("x_p" measured positive south along the 0° longitude meridian) and the y_p meridian (90°W or 270°E meridian). The International Reference Pole (IRP*) is the location of the terrestrial pole agreed upon by international committees. It results from observed values between 1900 and 1905. Deflections locate the IRP from the CEP. The motion is roughly a circular spiral about the IRP. The maximum amplitude of the variations is about 9 m in any direction. With the advent of the IAU 2000 Theory, the CEP is now the Celestial Intermediate Pole (CIP), whose motion is defined in the GCRS “by the motion of the Tisserand mean axis of the Earth with periods greater than two days”. The

* IERS was formerly the Bureau International de l’Heure, BIH, and the IRP was formerly the Conventional International Origin (CIO).
Tisserand mean axis is defined in Seidelmann (1982) as the mean surface geographic axis. Figure 3-24 shows the geometry for the effect of polar motion.

**TABLE 3-5. Pole Positions.** This table shows the coordinates of the North Pole over a few decades. Get current values for highly precise studies. All units are arcseconds ("). Remember that $0.1'' = 0.000\ 028^\circ$, which is about 3.09 m for the surface of the Earth. $x_p$ is measured along the $0^\circ$ longitude meridian while $y_p$ refers to the $90^\circ$ W longitude meridian.

<table>
<thead>
<tr>
<th>Year</th>
<th>Jan 1</th>
<th>Apr 1</th>
<th>Jul 1</th>
<th>Oct 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_p''$</td>
<td>$y_p''$</td>
<td>$x_p''$</td>
<td>$y_p''$</td>
</tr>
<tr>
<td>2000</td>
<td>0.043</td>
<td>0.378</td>
<td>0.075</td>
<td>0.346</td>
</tr>
<tr>
<td>2001</td>
<td>-0.073</td>
<td>0.398</td>
<td>0.091</td>
<td>0.490</td>
</tr>
<tr>
<td>2002</td>
<td>-0.177</td>
<td>0.294</td>
<td>-0.031</td>
<td>0.541</td>
</tr>
<tr>
<td>2003</td>
<td>-0.088</td>
<td>0.188</td>
<td>-0.133</td>
<td>0.436</td>
</tr>
</tbody>
</table>

**Figure 3-24. Transformation Geometry Due to Polar Motion.** This transformation takes into account the actual location of the Celestial Ephemeris Pole (CEP) over time. It moves from a Pseudo Earth fixed (PEF) system about the CEP, to the ITRF system and the IERS Reference Pole (IRP). The inset plot shows the IRP motion for about 20 years. Notice the slight “structure” to the variations over time.
**Fundamental Arguments**

The fundamental arguments have remained relatively consistent through the various IAU theories for reduction (1980, 1996, 2000). They are an integral part of the new IAU 2000 Theory, and are presented here for the IAU 2000 Theory. Be aware that slightly different values were used with the previous theories. There is an argument for the nutation, as well as an argument for the planetary nutation.

The nutation argument uses the mean anomalies (Sec. 2.2) for the Moon and Sun\(^*\), \(M_\odot, M_\odot\), the mean argument of latitude of the Moon, \(u_{M_\odot}\), measured on the ecliptic from the mean equinox of date, the mean elongation\(^\dagger\) from the Sun, \(D_\odot\), and the right ascension of the ascending node of the mean lunar orbit, \(\Omega_\odot\). \(\Omega_\odot\) is measured along the ecliptic from the mean equinox of date. These parameters are sometimes known as the Delaunay arguments, or the fundamental arguments, and they account for luni-solar nutation. Using \(r = 360^\circ\) and converting the mixed units from McCarthy (2003) to degrees,

\[
\begin{align*}
M_\odot &= 134.963 402 51^\circ + (1325^r + 198.867 560 5)T_{TDB} + 0.008 855 37T_{TDB}^2 + 1.4343 \times 10^{-5} T_{TDB}^3 \\
&\quad -6.797 \times 10^{-6} T_{TDB}^4 \\
M_\odot &= 357.529 109 18^\circ + (99^r + 359.050 291 1)T_{TDB} - 0.000 153 77T_{TDB}^2 - 3.8 \times 10^{-8} T_{TDB}^3 \\
&\quad -3.19 \times 10^{-9} T_{TDB}^4 \\
u_{M_\odot} &= 93.272 090 62^\circ + (1342^r + 82.017 457 7)T_{TDB} - 0.003 542 07T_{TDB}^2 + 2.88 \times 10^{-7} T_{TDB}^3 \\
&\quad + 1.16 \times 10^{-9} T_{TDB}^4 \\
D_\odot &= 297.850 195 47^\circ + (1236^r + 307.111 446 9)T_{TDB} - 0.001 769 67T_{TDB}^2 + 1.831 \times 10^{-6} T_{TDB}^3 \\
&\quad -8.80 \times 10^{-9} T_{TDB}^4 \\
\Omega_\odot &= 125.044 555 01^\circ - (5^r + 134.136 185 1)T_{TDB} + 0.002 075 67T_{TDB}^2 + 2.139 \times 10^{-6} T_{TDB}^3 \\
&\quad -1.650 \times 10^{-9} T_{TDB}^4
\end{align*}
\]

(3-51)

We can also include corrections for planetary effects on the nutation and the obliquity of the ecliptic. These arguments are the mean Heliocentric longitudes of the planets (\(\lambda_p\)), and the general precession in longitude (\(\rho_p\)).

\[
\begin{align*}
\lambda_{M_\odot} &= 252.250 905 494^\circ + 149,472.674 635 8T_{TDB} \\
\lambda_{M_\odot} &= 181.979 800 853^\circ + 58,517.815 674 8T_{TDB} \\
\lambda_{M_\odot} &= 100.466 448 494^\circ + 35,999.372 852 1T_{TDB} \\
\lambda_{M_\odot} &= 355.433 274 605^\circ + 19,140.299 314 T_{TDB}
\end{align*}
\]

\* The angles (\(M_\odot, M_\odot\)) are actually found using the Mean longitude (of the Sun and Moon) minus the longitude of periapsis of each object.

\dagger The elongation is the geocentric angle of a satellite, east or west (0 - 180°) of a planet (or of a planet from the Sun).
3.7 Transforming Celestial and Terrestrial Coordinates

3.7.1 IAU 2000 Reduction

The IAU 2000 Theory is presented for transformations between the body-fixed (ITRF), and the inertial (GCRF) frames (McCarthy, 2003). The original basis was the non-rotating origin (now the CEO). Guinot (1979) introduces the non-rotating sphere by describing its attachment to the stars, and its relation to a terrestrial sphere attached to the Earth. The reference system is realized by a great circle and a reference point. The motion of the pole is constrained so that “when [it] moves on the non-rotating sphere, the instantaneous system has no component of rotation around [the non-rotating pole]” Guinot (1979). Thus, the transformation is kinematically correct because it segregates the terrestrial and celestial motion of the pole from the Earth’s rotation (Seago and Vallado, 2000). This system is significant in that it eliminates the older distinctions of MOD, TOD, etc. This will be advantageous for numerical operations as less potential confusion will arise from the use of undocumented, intermediate coordinate frames.

Polar Motion (IAU 2000)

The method begins by applying polar motion. There is an additional rotation to account for the instantaneous prime meridian, $s’$—after January 1, 2003, it’s “necessary to provide an exact realization of the ‘instantaneous prime meridian’” (McCarthy, 2003). Notice that the pseudo-Earth Fixed (PEF) rotations are opposite from FK5, Eq. (3-66).

$$\hat{r}_{\text{PEF}} = \text{ROT3}(-s’)\text{ROT2}(x_p)\text{ROT1}(y_p)\hat{r}_{\text{ITRF}}$$

The main component of $s’$ is found using the average values for the Chandler wobble ($a_c = 0.26''$) and the annual wobble ($a_a = 0.12''$) of the pole. It will be less than 0.0004'' over the next century.

$$s’ = 0.0015'' \left( \frac{a_c^2}{1.2} + a_a^2 \right) T_{TT} \approx -0.000 047'' T_{TT}$$

The usual $x_p$ and $y_p$ values are sometimes augmented by tidal and nutation contributions. The combined rotation matrix from ITRF to PEF is:

$$\begin{bmatrix}
\cos(x_p)\cos(s’) & -\cos(y_p)\sin(s’) + \sin(y_p)\sin(x_p)\cos(s’) & -\sin(y_p)\sin(s’) - \cos(y_p)\sin(x_p)\cos(s’)
\cos(x_p)\sin(s’) & \cos(y_p)\cos(s’) + \sin(y_p)\sin(x_p)\sin(s’) & \sin(y_p)\cos(s’) - \cos(y_p)\sin(x_p)\sin(s’)
\sin(x_p) & -\sin(y_p)\cos(x_p) & \cos(y_p)\cos(x_p)
\end{bmatrix}$$
Earth Rotation Angle (IAU 2000)

The rotation of the Earth is taken into account by the Earth Rotation Angle, $\theta_{ERA}$—the angle between the CEO and the Terrestrial Ephemeris Origin, TEO (a reference meridian on the Earth) along the equator of the CIP. $\theta_{ERA}$ uses UT1 (McCarthy, 2003).

$$\theta_{ERA} = 280.460\,618\,375\,04^\circ + 360.985\,612\,288\,808(JD_{UT1} - 2,451,545.0)$$

$$= 2\pi(0.779\,057\,273\,624\,0 + 1.002\,737\,811\,911\,354\,48(JD_{UT1} - 2,451,545.0))$$ \hfill (3-52)

The transformation yields the intermediate reference frame of epoch. Notice the transformation for velocity.

$$\hat{\vec{r}}_{IRE} = \mathbf{ROT3}(-\theta_{ERA})\hat{\vec{r}}_{pef}$$

$$\hat{\vec{v}}_{IRE} = \mathbf{ROT3}(-\theta_{ERA})\hat{\vec{v}}_{ITRF} + \vec{\omega}_\oplus \times \hat{\vec{r}}_{pef}$$

Precession-Nutation (IAU 2000)

The coordinates of the CIP in the ICRS ($X$ and $Y$) are used to locate the GCRF position and velocity vectors (check the sign of the (1,2) term in future IERS updates).

$$\hat{\vec{r}}_{GCRF} = \begin{bmatrix} 1 - aX^2 & -aXY & X \\ -aXY & 1 - aY^2 & Y \\ -X & -Y & 1 - a(X^2 + Y^2) \end{bmatrix} \mathbf{ROT3}(s)\hat{\vec{r}}_{IRE}$$

where

$$a = \frac{1}{1 + \cos(d)} \approx \frac{0.5''}{1 + \cos(d)} = 0.125(X^2 + Y^2)$$

Notice here that $d$ is an argument that helps define the CIP in the ICRS.

McCarthy shows how to find the $X$ and $Y$ coordinates of the CIP with respect to the GCRS. Both expressions use the fundamental arguments to form the trigonometric term.

$$a_{f_1} = a_{0x1} M_\xi + a_{0x2} M_\zeta + a_{0x3} M_\phi + a_{0x4} M_\theta + a_{0x5} M_\psi + a_{0x6} M_\chi + a_{0x7} M_\lambda + a_{0x8} M_\mu + a_{0x9} M_\nu + a_{0x10} M_\alpha + a_{0x11} M_\beta + a_{0x12} M_\gamma + a_{0x13} M_\delta + a_{0x14} M_\omega$$

Be careful on the order of these coefficients and terms. Notice that the coefficients of the fundamental arguments have “0”, “x”, “1”, and “i” subscripts. The “0” indicates the index of the terms which corresponds to a summation quantity (1-5), “x” indicates the parameter under consideration ($X$, $Y$, $s$), “1” indicates the term that the coefficient applies to (1-14), and “i” is the index for the number of terms in the summation (1 - ...). For the main coefficients (capital letters), the first “0” is deleted and a “s” or “c” is added to indicate that it is with the sine or cosine expression. Also, although the last
expression is a summation of only one term, future revisions may include additional terms, and the existing values may change slightly.

\[-0.198\,620\,547\,T_{TDB}^3 - 0.000\,046\,054\,T_{TDB}^4 + 0.000\,005\,987\,T_{TDB}^5\]

\[\sum_{i=1}^{136} A_{x\alpha i} \sin \{a_{p_i}\} + A_{x\gamma i} \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{23} A_{x1\alpha i} T_{TDB} \sin \{a_{p_i}\} + A_{x1\gamma i} T_{TDB} \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{36} A_{x2\alpha i} T_{TDB}^2 \sin \{a_{p_i}\} + A_{x2\gamma i} T_{TDB}^2 \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{4} A_{x3\alpha i} T_{TDB}^3 \sin \{a_{p_i}\} + A_{x3\gamma i} T_{TDB}^3 \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{1} A_{x4\alpha i} T_{TDB}^4 \sin \{a_{p_i}\} + A_{x4\gamma i} T_{TDB}^4 \cos \{a_{p_i}\}\]

\[Y = -0.006\,950\,789 - 0.025\,381\,999\,T_{TDB} - 22.407\,250\,999\,T_{TDB}^2\]

\[+ 0.001\,842\,287\,T_{TDB}^3 + 0.001\,113\,067\,T_{TDB}^4 + 0.000\,000\,999\,T_{TDB}^5\]

\[\sum_{i=1}^{962} A_{y\alpha i} \sin \{a_{p_i}\} + A_{y\gamma i} \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{277} A_{y1\alpha i} T_{TDB} \sin \{a_{p_i}\} + A_{y1\gamma i} T_{TDB} \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{30} A_{y2\alpha i} T_{TDB}^2 \sin \{a_{p_i}\} + A_{y2\gamma i} T_{TDB}^2 \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{5} A_{y3\alpha i} T_{TDB}^3 \sin \{a_{p_i}\} + A_{y3\gamma i} T_{TDB}^3 \cos \{a_{p_i}\}\]

\[\sum_{i=1}^{1} A_{y4\alpha i} T_{TDB}^4 \sin \{a_{p_i}\} + A_{y4\gamma i} T_{TDB}^4 \cos \{a_{p_i}\}\]
The value of \( s \) “provides the position of the CEO on the equator of the CIP corresponding to the kinematical definition of the [non-rotating origin] in the GCRS when the CIP is moving with respect to the GCRS between the reference epoch and the epoch due to precession and nutation” (McCarthy 2003). It is found by

\[
\begin{align*}
&\sum_{i=1}^{1} \sum_{j=1}^{TDB} A_{s,1i} TDB \sin \{a_{p_i}\} + A_{s,2i} TDB \cos \{a_{p_i}\} \\
&\sum_{i=1}^{25} \sum_{j=1}^{TDB} A_{s,2i} TDB^2 \sin \{a_{p_i}\} + A_{s,3i} TDB^2 \cos \{a_{p_i}\} \\
&\sum_{i=1}^{4} \sum_{j=1}^{TDB} A_{s,3i} TDB^3 \sin \{a_{p_i}\} + A_{s,4i} TDB^3 \cos \{a_{p_i}\} \\
&\sum_{i=1}^{1} \sum_{j=1}^{TDB} A_{s,4i} TDB^4 \sin \{a_{p_i}\} + A_{s,5i} TDB^4 \cos \{a_{p_i}\} \\
&0.000 \ 001 \ 71^\circ TDB \sin (\Omega - \delta) + 0.000 \ 003 \ 57 TDB \cos (2 \Omega - \delta) \\
&0.000 \ 743 \ 53 TDB^2 \sin (\Omega - \delta) + 0.005 \ 056 \ 91 TDB^2 \sin (2 (u_M - D - \Omega - \delta))
\end{align*}
\]

McCarthy (2003) discusses another method to find \( x \) and \( y \) using the classical equinox-based transformation, but I won’t detail it here because there are two additional methods specifically tailored to this approach.

**Classical Equinox Based Transformations with IAU 2000**

The previous reductions may also be expressed in the classical equinox-based notation. There are two levels of accuracy, the IAU 2000a and the IAU 2000b. At the time of this writing, the theory had just been released, so I won’t detail these methods because there are still some parameters that may change. I simply present the current estimates for the parameters used in this approach so you can compare with the former FK5 theory.

These methods use the obliquity of the ecliptic, \( \epsilon \). We can express the effect on \( \dot{\epsilon} \) as a mean function of time due to the precession effects on the Earth. Thus, a time varying expression is
3.7 Transforming Celestial and Terrestrial Coordinates

\[ \varepsilon = 84.381.448^\circ - 46.84024 T_{TDB} - 0.000 59 T_{TDB}^2 + 0.001 813 T_{TDB}^3 \]
\[ \varepsilon = 23.439 291^\circ - 0.013 011 1 T_{TDB} - 1.64 \times 10^{-7} T_{TDB}^2 + 5.04 \times 10^{-7} T_{TDB}^3 \]  

(3-53)

where \( T_{TDB} \) is the Julian centuries from J2000.*

The nutation (\( \Delta \Psi \)) and obliquity (\( \Delta \epsilon \)) are of similar form to previous reduction theories. As with the classical FK5 transformation, there are contributions from nutation and luni-solar terms. Be careful to use the correct formulae, coefficients, and units. Notice there are no “0” subscripts as there are only one set of coefficients in each expression. Also, the argument (\( x \)) is now a \( n \) for nutation (the IAU 2000b theory has only 77 terms).

\[ \Delta \Psi_{2000n} = \sum_{i=1}^{678} \left\{ A_{p_i} + A_{pl-i} T_{TDB} \right\} \sin(a_{p_i}) + \left\{ A_{pc_i} + A_{pc-l-i} T_{TDB} \right\} \cos(a_{p_i}) \]
\[ \Delta \epsilon_{2000n} = \sum_{i=1}^{678} \left\{ A_{ei_i} + A_{ei-l-i} T_{TDB} \right\} \sin(a_{p_i}) + \left\{ A_{esi_i} + A_{esi-l-i} T_{TDB} \right\} \cos(a_{p_i}) \]  

(3-54)

\[ a_{p_i} = a_{n1} M\xi + a_{n2} M\phi + a_{n3} u M\xi + a_{n4} D\phi + a_{n5} \Omega \xi \]

The planetary contribution series expression is (the IAU 2000b theory uses constant values, \( \Delta \Psi_{2000pl} = -0.000 135^\circ \) and \( \Delta \epsilon_{2000pl} = 0.000 388^\circ \))

\[ \Delta \Psi_{2000pl} = \sum_{i=1}^{687} A_{pi} \sin(a_{p_i}) + A_{pc_i} \cos(a_{p_i}) \]
\[ \Delta \epsilon_{2000pl} = \sum_{i=1}^{687} A_{ei_i} \cos(a_{p_i}) + A_{esi_i} \sin(a_{p_i}) \]

\[ a_{p_i} = a_{p11} M\xi + a_{p12} M\phi + a_{p13} u M\xi + a_{p14} D\phi + a_{p15} \Omega \xi + a_{p6} \lambda_{M\zeta} + a_{p7} \lambda_{M\zeta} + a_{p8} \lambda_{M\zeta} + a_{p9} \lambda_{M\zeta} + a_{p10} \lambda_{M\zeta} + a_{p11} \lambda_{M\zeta} + a_{p12} \lambda_{M\zeta} + a_{p13} \lambda_{M\zeta} + a_{p14} p_a \]

* Notice that the coefficient of the first term in Eq. (3-53) is the approximate decrease in the obliquity of the ecliptic resulting from planetary precession (0.013 011° [47"] per century)
The combined result uses the planetary terms, and the complete matrix is the same as shown on page 223.

\[
\Delta \Psi = \Delta \Psi_{2000n} + \Delta \Psi_{2000p} \\
\Delta \epsilon = \Delta \epsilon_{2000n} + \Delta \epsilon_{2000p} \tag{3-55}
\]

Similar to Eq. (3-57), the precession formula for the IAU 2000 Theory is

\[
\zeta = 2.597 \, 617 \, 6'' + 2306.080 \, 950 \, 6T_{\text{DDB}} + 0.301 \, 901 \, 5T_{\text{DDB}}^2 + 0.017 \, 966 \, 3T_{\text{DDB}}^3 \\
- 0.000 \, 032 \, 7T_{\text{DDB}}^4 - 0.000 \, 000 \, 2T_{\text{DDB}}^5 \\
\Theta = 2004.191 \, 747 \, 6T_{\text{DDB}} - 0.426 \, 935 \, 3T_{\text{DDB}}^2 - 0.041 \, 825 \, 1T_{\text{DDB}}^3 \\
- 0.000 \, 060 \, 1T_{\text{DDB}}^4 - 0.000 \, 000 \, 1T_{\text{DDB}}^5 \\
z = 2.597 \, 617 \, 6'' + 2306.080 \, 322 \, 6T_{\text{DDB}} + 1.094 \, 779 \, 0T_{\text{DDB}}^2 + 0.018 \, 227 \, 3T_{\text{DDB}}^3 \\
+ 0.000 \, 047 \, 0T_{\text{DDB}}^4 - 0.000 \, 000 \, 3T_{\text{DDB}}^5
\]

The Greenwich Mean Sidereal Time has the following form in the IAU2000 Theory.

\[
\sigma_{\text{GMST}} - \sigma_{\text{ERA}} = 0.014 \, 300 + 4914.131 \, 001 \, \Delta \Psi + 1.590 \, 011 \, \Delta \Psi \cos(\epsilon) \\
- 0.000 \, 093 \, 44T_{\text{TT}}^3 + 0.000 \, 018 \, 827T_{\text{TT}}^4 + \Delta \Psi \cos(\epsilon) \\
- 0.000 \, 000 \, 87T_{\text{TT}}^5 \sin(a) - \sum_{i=1}^{33} A_{g_{x_{ij}}} \sin(a_{p_j}) + A_{g_{c_{ij}}} \cos(a_{p_j}) \\
- \sum_{i=1}^{33} A_{g_{x_{ij}}} T_{\text{TT}} \sin(a_{p_j}) + A_{g_{c_{ij}}} T_{\text{TT}} \cos(a_{p_j})
\]

**Implementing the IAU 2000 Theory**

We can visualize the transformation as follows. It’s important to realize that the former distinctions of “date” and “epoch” are essentially removed with the new theory. Also notice the matrix inverse for [PN].

**Algorithm 23: IAU 2000**

\[
\begin{bmatrix}
\hat{r}_{\text{GCRF}} \\
\hat{v}_{\text{GCRF}}
\end{bmatrix}
\begin{bmatrix}
\text{yr, mo, day, UTC, } \Delta UT1, \Delta AT, x_p, y_p
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\hat{r}_{\text{ITRF}} \\
\hat{v}_{\text{ITRF}}
\end{bmatrix}
\]

**CONVTIME** \((\text{yr, mo, day, UTC, } \Delta UT1, \Delta AT) \Rightarrow \text{UT1, TAI, TT, } T_{\text{ITRF}}, T_{\text{TT}}\)

Find values for \(a, X, Y, s', s, \theta_{\text{ERA}}\)
Figure 3-25. **IAU 2000 Transformation**. This figure shows the transformation of a vector in the body fixed (ITRF) frame to the inertial (GCRF) frame. This two-way conversion is necessary for many orbit determination problems.

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
1 - aX^2 & -aXY \\
-aXY & 1 - aY^2
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix}
\]

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} = \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix}^{-1}
\]

\[
\begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} = \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix}^{-1}
\]

\[
\begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} = \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
\cdot \\
\cdot
\end{bmatrix}^{-1}
\]

**Example 3-14. Performing an IAU2000 Reduction**

**GIVEN:**

\[
\begin{array}{l}
\hat{\mathbf{r}}_{GCRF} = 5102.508 \hat{I} + 6123.011 \hat{J} + 6378.136 \hat{K} \text{ km} \\
\hat{\mathbf{v}}_{GCRF} = -4.743 \ 220 \ 16 \hat{I} + 0.790 \ 536 \ 50 \hat{J} + 5.533 \ 756 \ 573 \hat{K} \text{ km/s}
\end{array}
\]
FIND: $\hat{r}_{ITRF}, \hat{v}_{ITRF}$ on April 6, 2004, 07:51:28.386 009 UTC

Calculate the time quantities before starting the problem. Get $\Delta T$ (32.0s) directly from the Astronomical Almanac (2004, K9). From the IERS’s EOPCO4 final data, find $\Delta UT_1 = -0.439 961 9s$, $\chi_p = -0.140 682^\circ$ and $\gamma_p = 0.333 309^\circ$, and $\delta\Delta \Psi = -0.052 195^\circ$ and $\delta\Delta \lambda = -0.003 875^\circ$. Although general practice interpolates these values with a spline, don’t interpolate these values for this example.

$UT_1 = UTC + \Delta UT = 07:51:28.386 009 - 0.439 969 9 = 07:51:27.946 047$

$TAI = UTC + \Delta AT = 07:51:28.386 009 + 32^s = 07:52.086 009$

$TT = TAI + 32.184^s = 07:52.086 009 + 32.184^s = 07:52.327 009$

$DMY HMS \Rightarrow JD_{TT} = 2,453,101.828 154 745$ (Be sure to keep all digits for accuracy.)

$T_{TT} = \frac{JD_{TT} - 2,451,545.0}{36,525} = \frac{2,453,101.828 154 745 - 2,451,545.0}{36,525} = 0.042 623 631 9$

Determine the angles for the Earth’s nutation ($r = 360^\circ$).

$M_\odot = 134.963 402 51^\circ + (1325r + 198.867 560 5)T_{TDB} + 0.008 855 3T_{TDB}^2$

+ $1.4343 \times 10^{-5} T_{TDB}^3 - 6.797 \times 10^{-6} T_{TDB}^4$

$M_\odot = 357.529 109 18^\circ + (99r + 359.050 291 1)T_{TDB} - 0.000 153 7T_{TDB}^2 - 3.8 \times 10^{-8} T_{TDB}^3$

$-3.19 \times 10^{-4} T_{TDB}^4$

$\nu M_\odot = 93.272 090 62^\circ + (1342r + 82.017 457 7)T_{TDB} - 0.003 542 07T_{TDB}^2 + 2.88 \times 10^{-7} T_{TDB}^3$

+ $1.16 \times 10^{-9} T_{TDB}^4$

$D_\odot = 297.850 195 47^\circ + (1236r + 307.111 446 9)T_{TDB} - 0.001 769 6T_{TDB}^2$

+ $1.83 \times 10^{-6} T_{TDB}^3 - 8.8 \times 10^{-9} T_{TDB}^4$

$\Omega_\odot = 125.044 555 01^\circ - (5r + 134.136 185 1)T_{TDB} + 0.002 075 6T_{TDB}^2 + 2.13 \times 10^{-6} T_{TDB}^3$

$-1.65 \times 10^{-8} T_{TDB}^4$

$M_\odot = 314.912 287 3^\circ, M_\odot = 91.939 376 9^\circ,$

$\nu M_\odot = 169.097 004 3^\circ, D_\odot = 196.751 642 8^\circ,$

$\Omega_\odot = 42.604 650 0^\circ$

Next, the planetary nutation values are found.
3.7 Transforming Celestial and Terrestrial Coordinates

Now find the coordinates \( x, y, \) and \( s \). I show only the final answers due to the length of the calculations. In radians,
\[
\begin{align*}
\lambda_{\text{M} \xi} &= 252.250\,905.494^\circ + 149.472.674\,635\,8^\circ T_{\text{TDB}} = 143.319\,167^\circ \\
\lambda_{\text{M} \eta} &= 181.979\,800.853^\circ + 58.517.815\,674\,8^\circ T_{\text{TDB}} = 156.221\,635^\circ \\
\lambda_{\text{M} \phi} &= 100.466\,448\,494^\circ + 35.999.372\,852\,1^\circ T_{\text{TDB}} = 194.890\,465^\circ \\
\lambda_{\text{M} \psi} &= 355.433\,274\,605^\circ + 19.140.299\,314\,T_{\text{TDB}} = 91.262\,347^\circ \\
\lambda_{\text{M} \lambda} &= 34.351\,483\,900^\circ + 3.034.905\,674\,64^\circ T_{\text{TDB}} = 156.710\,186^\circ \\
\lambda_{\text{M} \beta} &= 50.077\,471\,399\,8^\circ + 1.222.113\,794\,04^\circ T_{\text{TDB}} = 102.816\,400^\circ \\
\lambda_{\text{M} \delta} &= 314.055\,005\,137^\circ + 428.466\,998\,313\,T_{\text{TDB}} = 332.317\,825^\circ \\
\lambda_{\text{M} \phi} &= 304.348\,665\,499^\circ + 218.486\,200\,208\,T_{\text{TDB}} = 313.661\,341^\circ \\
P_a &= 1.396\,971\,372\,14^\circ T_{\text{TDB}} + 0.000\,308\,66^\circ T_{\text{TDB}}^2 = 0.059\,545^\circ \\
\end{align*}
\]

Now find the coordinates \( x, y, \) and \( s \). I show only the final answers due to the length of the calculations. In radians,
\[
\begin{align*}
x &= 0.022\,370\,0, y = 0.002\,020\,5, s = -0.000\,000\,8 \\
a &= 0.000\,138\,889.
\end{align*}
\]

When performing transformations from inertial to fixed, you cannot simply take the transpose of the precession-nutation matrix because the order of rotations will be incorrect for the combined matrix \([\text{PN}]\). McCarthy (2003) shows initial rotations \( \text{ROT3}\left(-E\right) \text{ROT1}\left(-d\right) \text{ROT3}\left(E\right) \), and the assembled matrix, but no suitable expressions for \( E \) and \( d \), thus a traditional matrix inverse for \([\text{PN}]\) is required when converting from inertial to fixed. The rotations produce the intermediate reference epoch.

\[
\begin{align*}
\hat{\lambda}_{\text{IBE}} &= 5100.007\,639\,3 \hat{I} + 6122.776\,411\,5 \hat{J} + 6380.343\,827\,1 \hat{K} \text{ km} \\
\hat{v}_{\text{IRE}} &= -4.745\,388\,938 \hat{I} + 0.790\,332\,038 \hat{J} + 5.531\,929\,087 \hat{K} \text{ km/s}
\end{align*}
\]

To find the Earth rotation angle, we use Eq. (3-52). \( \text{JD}_{\text{UT1}} = 2,453,101.827\,406\,783 \).
\(
\theta_{\text{ERA}} = 312.755\,282\,9^\circ
\)

After doing the rotation, you’ll get these PEF vectors:
\[
\begin{align*}
\hat{\lambda}_{\text{PEF}} &= -1033.475\,031\,3 \hat{I} + 7901.290\,924\,0 \hat{J} + 6380.343\,827\,1 \hat{K} \text{ km} \\
\hat{v}_{\text{PEF}} &= -3.225\,632\,747 \hat{I} - 2.872\,455\,223 \hat{J} + 5.531\,929\,087 \hat{K} \text{ km/s}
\end{align*}
\]

Finally, include the effects of polar motion. Convert to radians and find the final ECEF coordinates.
\[
\begin{align*}
\hat{\lambda}_{\text{ITRF}} &= -1033.479\,383\,0 \hat{I} + 7901.295\,275\,8 \hat{J} + 6380.356\,595\,3 \hat{K} \text{ km} \\
\hat{v}_{\text{ITRF}} &= -3.225\,636\,520 \hat{I} - 2.872\,451\,450 \hat{J} + 5.531\,924\,446 \hat{K} \text{ km/s}
\end{align*}
\]

3.7.2 FK5 Reduction

Although this is no longer the current theory, I’ve included a discussion adapted from McCarthy (1996), Frank (1984), Seidelmann (1992, 99–145), the Astronomical Almanac (1992, B18–B19), NIMA (2000), and the Defense Mapping Agency (1987b, A15–A31) because many programs still reference this system.
Precession (FK5)

The 1976 IAU Precession theory models the transformation of systems with certain angles shown in Fig. 3-26. The transformation is obtained by recognizing that a common point exists between the mean equator at J2000 and the ecliptic of date. This permits a motion along the mean equator at J2000, a translation to the mean equator of date, and a final motion to the location of the equinox of date. The process converts a vector in the FK5 system, having a mean equinox at J2000, to a vector having the mean equinox of date (MOD). The angles $\zeta$, $\Theta$, and $\varpi$ represent the combined effects of general precession. Fortunately, these angles are secular values and are easily calculated (Seidelmann, 1992, 104) and Astronomical Almanac (1984, S19). Eq. (3-56) presents the formulas necessary to determine the precession angles. We use a base epoch (J2000 in this case), an arbitrary fixed epoch, $JD_{\text{start}}$, and the epoch of date, $JD_{\text{end}}$. Then, we use these Julian dates (TDB $\equiv$ TT is the suggested practice after 1994) to determine the number of Julian centuries required for the formulas.
3.7 Transforming Celestial and Terrestrial Coordinates

\[ T_o = \frac{JD_{start} - J2000}{36,525} \quad T_\Delta = \frac{JD_{end} - JD_{start}}{36,525} \]

\[ \zeta = (2306.2181'' + 1.396 \, 56 \, T_o - 0.000139 \, T^2_o) \Delta^2 + 0.301 \, 88 - 0.000 \, 344 \, T_o \Delta^2 + 0.017 \, 998 \, T^3_o \Delta \] (3-56)

\[ \Theta = (2004.3109'' - 0.853 \, 307 \, T_o - 0.000 \, 217 \, T^2_o) \Delta^2 + (-0.42665 - 0.000 \, 0217 \, T_o) \Delta^2 - 0.041 \, 833 \, T^3_o \Delta \]

\[ z = (2306.2181'' + 1.396 \, 56 \, T_o - 0.000139 \, T^2_o) \Delta^2 + (1.094 \, 68 + 0.000 \, 066 \, T_o) \Delta^2 + 0.018 \, 203 \, T^3_o \Delta \]

Eq. (3-56) allows direct transformation from a starting to ending date, without having to do an intermediate transformation to the J2000 epoch. In many cases, however, the fixed epoch and the base epoch are the same \((T_o = 0)\), and we can simplify the formulas:

\[ \zeta = 2306.2181'' \Delta T_{TDB} + 0.301 \, 88 \Delta^2 T^2_{TDB} + 0.017 \, 998 \Delta^3 T^3_{TDB} \]

\[ \Theta = 2004.3109'' \Delta T_{TDB} - 0.426 \, 65 \Delta^2 T^2_{TDB} - 0.041 \, 833 \Delta^3 T^3_{TDB} \]

\[ z = 0.640 \, 616 \, 1^\circ \Delta T_{TDB} + 0.000 \, 083 \, 9 \Delta^2 T^2_{TDB} + 5.0 \times 10^{-6} \Delta^3 T^3_{TDB} \] (3-57)

\[ \Theta = 0.556 \, 753 \, 0^\circ \Delta T_{TDB} - 0.000 \, 118 \, 5 \Delta^2 T^2_{TDB} - 1.16 \times 10^{-5} \Delta^3 T^3_{TDB} \]

\[ z = 0.640 \, 616 \, 1^\circ \Delta T_{TDB} + 0.000 \, 304 \, 1 \Delta^2 T^2_{TDB} + 5.1 \times 10^{-6} \Delta^3 T^3_{TDB} \]

\( T_{TDB} [\text{Eq. (3-40)}] \) represents the number of Julian centuries of TDB from the base epoch (J2000), but common practice uses \( T_{TT} \). The order of the rotations is important, and for precession, we have identical relations for position and velocity.

\[ \hat{r}_{mod} = \text{ROT}3(-z)\text{ROT}2(\Theta)\text{ROT}3(-\zeta)\hat{r}_{FK5} \] (3-58)

The complete rotation matrix for transformations from FK5 to MOD is

\[
\begin{bmatrix}
\cos(\Theta)\cos(z)\cos(\zeta) - \sin(z)\sin(\zeta) & -\sin(z)\cos(\Theta)\cos(\zeta) - \sin(z)\sin(\zeta) & -\sin(\Theta)\cos(z) - \sin(\Theta)\sin(\zeta) \\
\sin(z)\cos(\Theta)\sin(\zeta) + \sin(z)\cos(z) & \sin(z)\sin(\Theta)\sin(\zeta) + \sin(z)\cos(z) & 0 \\
\sin(\Theta)\cos(z) & -\sin(\Theta)\sin(z) & \cos(\Theta)
\end{bmatrix}
\]

**Nutation (FK5)**

The next transformation accounts for the periodic effects contributed primarily by the Moon. The primary expression is from the 1980 IAU Theory of Nutation. We determine the primary variation from a trigonometric series of 106 terms. The number of terms is driven by the arc-second accuracy requirements at the development time. The result is periodic, rather than secular like the precession effect. Adding these effects transforms the mean equator of date to the true equator of date, TOD. Figure 3-27 shows both the mean and true obliquity of the ecliptic values.

The first step is to determine the mean obliquity of the ecliptic. FK5 defines the obliquity of the ecliptic (at J2000) as \( \dot{\epsilon} = 23^\circ26'21.448'' = 23.439 \, 291^\circ \). We use Eq. (3-53) and the angles from Eq. (3-51). Note that the second term is different here.
Once we find these values, we evaluate a trigonometric series to find the nutation in longitude, \(\Delta W\), and the nutation in obliquity, \(\Delta \varepsilon\). Table D-6 lists sample coefficients for the integers \((a_{i1} - a_{i5})\) and the real coefficients \((A_i - D_i)\).

\[
\tilde{\varepsilon} = 84.381.448'' - 46.8150 T_{TDB} - 0.000 59 T_{TDB}^2 + 0.001 813 T_{TDB}^3
\]
\[
\tilde{\varepsilon} = 23.439 291° - 0.013 004 2 T_{TDB} - 1.64 \times 10^{-7} T_{TDB}^2 + 5.04 \times 10^{-7} T_{TDB}^3
\]

Once we find these values, we evaluate a trigonometric series to find the nutation in longitude, \(\Delta \Psi\), and the nutation in obliquity, \(\Delta \varepsilon\). Table D-6 lists sample coefficients for the integers \((a_{i1} - a_{i5})\) and the real coefficients \((A_i - D_i)\).

\[
\Delta \Psi_{1980} = \sum_{i=1}^{106} (A_{p_i} + A_{p1_i} T_{TDB}) \sin \{a_{p_i}\}
\]
\[
\Delta \varepsilon_{1980} = \sum_{i=1}^{106} (A_{e_i} + A_{e1_i} T_{TDB}) \cos \{a_{p_i}\}
\]
\[
a_p = a_{i1} \mu_{\psi} + a_{i2} M_{\odot} + a_{i3} \mu M_{\psi} + a_{i4} D_{\odot} + a_{i5} \Omega_{\psi}
\]
Despite the effort to model nutation with these calculations, discrepancies exist when processing *Very Long Baseline Interferometry* (VLBI) and *Lunar Laser Ranging* (LLR) data. Thus, if compatibility with the GCRF is needed, you use the 106 term series, and add EOP corrections ($\delta \Delta \epsilon$, $\delta \Delta \Psi$) to bring the nutation parameters closer to the GCRF values. The combined update for the obliquity of the ecliptic and the nutation are

$$\Delta \Psi = \Delta \Psi_{1980} + \delta \Delta \Psi \quad \Delta \epsilon = \Delta \epsilon_{1980} + \delta \Delta \epsilon$$

(3-61)

The EOP corrections are *not* used for applications referencing FK5.

$$\epsilon = \tilde{\epsilon} + \Delta \epsilon$$

(3-62)

Fig. 3-27 shows the transformation. The rotations needed to rigorously transform coordinates for nutation are (velocity is transformed the same as position)

$$\hat{\mathbf{r}}_{\text{tod}} = \text{ROT}1\left(-\epsilon\right)\text{ROT}3\left(-\Delta \Psi\right)\text{ROT}1\left(\tilde{\epsilon}\right)\hat{\mathbf{r}}_{\text{mod}}$$

(3-63)

The complete rotation matrix for transformations from MOD to TOD is

$$\begin{bmatrix}
\cos(\Delta \Psi) & -\sin(\Delta \Psi) \cos(\epsilon) & -\sin(\Delta \Psi) \sin(\epsilon) \\
\sin(\Delta \Psi) \cos(\epsilon) & \cos(\epsilon) \cos(\Delta \Psi) + \sin(\epsilon) \sin(\epsilon) & \sin(\epsilon) \cos(\epsilon) \cos(\Delta \Psi) - \sin(\epsilon) \cos(\epsilon) \\
\sin(\epsilon) \sin(\Delta \Psi) & \sin(\epsilon) \cos(\Delta \Psi) \cos(\epsilon) - \sin(\epsilon) \cos(\epsilon) & \sin(\epsilon) \sin(\epsilon) \cos(\Delta \Psi) + \cos(\epsilon) \cos(\epsilon)
\end{bmatrix}$$

**Sidereal Time (FK5)**

The third transformation is responsible for most of the time dependency of the vector components. This process is usually coupled with polar motion. It transforms the nonrotating true of date frame to a rotating coordinate system. It aligns the $I$-axis with the current site/satellite location. This motion addresses the variability in sidereal time and requires the Greenwich *apparent* sidereal time because the rotating frame references the Earth’s actual equator of date, which is the true equator. Greenwich apparent sidereal time is measured on the true equator relative to the true equinox, and Greenwich mean sidereal time is measured along the true equator relative to the mean equinox. The difference between these two times is called the equation of the equinoxes. This quantity is really the accumulated precession and nutation in right ascension. It’s obtained by projecting the difference between the mean and true equinoxes onto the true equator—thus a simple formula for the *equation of the equinoxes* is $\Delta \Psi \cos(\tilde{\epsilon})$, which we will see in the approximate form of the nutation matrix*. We use the equation of the equinoxes to convert between apparent and mean sidereal times. We can easily derive this relation from the right triangle having a central angle $\tilde{\epsilon}$. Be aware that before January 1, 1997, the last two terms (using $\Omega_{e}$) were not used (less than about 1 m). These terms make the (newer) kinematic and (older) geometric definitions more consistent. From Fig. 3-27, we find the Greenwich apparent sidereal time, $\theta_{\text{ASP}}$, using the Greenwich mean sidereal time [Eq. (3-45) or Eq. (3-47) and Eq. (3-44)], plus the equation of the equinoxes:

* Because the angle is actually measured along the mean equator and then projected onto the true equator, we use the mean obliquity of the ecliptic.
\[ EQ_{\text{equinox}} = \Delta \Psi \cos(\dot{\epsilon}) + 0.002\ 64'' \sin(\Omega_g) + 0.000\ 063 \sin(2\Omega_g) \]

\[ \theta_{\text{AST}} = \theta_{\text{GMST}} + EQ_{\text{equinox}} \]  

Notice the additional term for velocity transformations. The rotations are

\[ \hat{r}'_{\text{PEF}} = \text{ROT3}(\theta_{\text{AST}}) \hat{r}_{\text{tod}} \]

\[ \hat{v}'_{\text{PEF}} = \text{ROT3}(\theta_{\text{AST}}) \hat{v}_{\text{tod}} - \hat{\omega}_\oplus \times \hat{r}'_{\text{PEF}} \]  

**Polar Motion (FK5)**

To do this transformation rigorously involves two rotations. For PEF to ITRF,

\[ \hat{r}'_{\text{ITRF}} = \text{ROT2}(-x_p) \text{ROT1}(-y_p) \hat{r}'_{\text{PEF}} \left[ \begin{array}{ccc} \cos(x_p) & \sin(x_p) & \sin(x_p) \cos(y_p) \\ -\sin(x_p) \cos(y_p) & \cos(x_p) & -\sin(x_p) \cos(y_p) \\ -\sin(y_p) \cos(x_p) & \sin(y_p) \sin(x_p) & \cos(x_p) \cos(y_p) \end{array} \right] \]  

(3-66)

Because these values are very small—typically only a few meters in each direction—we often avoid the trigonometry for a rigorous coordinate transformation. Instead, we use the trigonometric small-angle approximation to simplify the results. The resultant matrix (with \(x_p\) and \(y_p\) values in radians) takes the form

\[ \hat{r}'_{\text{ITRF}} \approx \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{bmatrix} \hat{r}'_{\text{PEF}} \]  

(3-67)

Be aware of two observations: (1) The change in values over about one week is significant enough to require updates, assuming you need the correction at all, and (2) we can’t predict values with high accuracy over very long periods because the motion isn’t fully understood.

The variation in the CEP’s exact location also causes a slight shift in terrestrial coordinates (from the ITRF) of latitude and longitude (considered as mean values in this case). To estimate the offset, Seidelmann (1992, 140) provides these equations:

\[ \Delta \phi_{gd} \approx x_p \cos(\bar{\lambda}) - y_p \sin(\bar{\lambda}) \]

\[ \Delta \lambda \approx \{x_p \sin(\bar{\lambda}) - y_p \cos(\bar{\lambda})\} \tan(\bar{\phi}) \]

**Summary (FK5)**

We can apply the transformation equations in either direction, recognizing that the order of rotations is important for specific steps of the transformation. To transform completely between the celestial (FK5-J2000) and the terrestrial (ITRF) frames, we’ll need to do four separate groups of rotations and conversions: precession, nutation, sidereal time, and polar motion. Using the matrices from the previous sections,
3.7 Transforming Celestial and Terrestrial Coordinates

For the most precise calculations, we must do each of these transformations. Less accurate calculations sometimes use just the transformation for sidereal time. This option is often exercised and seldom documented!

The velocity transformation is similar to that already presented, except we change the sidereal time rotation, \([ST]\), to reflect the angular velocity of the Earth with respect to a Newtonian-inertial frame (Seago and Vallado, 2000). We use the time derivative of the stellar angle [Eq. (3-52)] to define the rotation rate of the Earth.

\[
\omega_\oplus = 7.292\,115\,146\,706\,98\times 10^{-5}\left\{ 1 - \frac{LOD}{86400} \right\}
\]  

(3-68)

where **Length of Day**, LOD is the instantaneous rate of change (in seconds) of UT1 with respect to a uniform time scale (UTC or TAI). LOD is maintained by the IERS. Although \(\omega_\oplus\) does not reference an inertial equinox, the difference is small for most satellite applications. However, the difference between \(\omega_\oplus\) and \(\omega_\oplus\) [Eq. (3-38)] is about as large as the average effect of LOD (~10^{-11} rad/sec). Thus, applications that use LOD will likely require \(\omega_\oplus\) in lieu of \(\omega_\oplus\). Abbreviating the rotations, we have

\[
\hat{v}_{ITRF} = [PM][ST][NUT][PREC]\hat{v}_{J2000} - \omega_\oplus \times \hat{r}_{PEF}
\]  

(3-69)

where \(ST =\) sidereal time, \(PM =\) polar motion, \(NUT =\) nutation, \(PREC =\) precession.

By using \(\hat{r}_{PEF}\), we correct the velocity before the final transformation for polar motion. The negative sign in the cross product occurs because the transformation is proceeding from the inertial system to the noninertial, Earth-fixed system. The literature contains alternate notations. Using the FK5 position vector and the time rate of change of the rotation matrix for sidereal time (because it’s the only parameter that varies much) gives us the same results [see also Eq. (3-25)]:

\[
\hat{v}_{ITRF} = [PM][ST][NUT][PREC]\hat{r}_{J2000} + [PM][ST][NUT][PREC]\hat{v}_{J2000}
\]

\[
[ST] = \begin{bmatrix}
-\omega_\oplus \sin(\theta_{AST}) & \omega_\oplus \cos(\theta_{AST}) & 0 \\
-\omega_\oplus \cos(\theta_{AST}) & -\omega_\oplus \sin(\theta_{AST}) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Although this approach yields identical answers, it doesn’t show the dependence on the rotating coordinate systems. If the acceleration is also needed, use Eq. (3-24) similar to the use of Eq. (3-23) in Eq. (3-69) above. Finally, to reverse the transformation, be careful of the order of operations and remember to use transposes of rotation matrices.
Implementing FK5 Reductions

Algorithm 24, supports FK5 reductions to and from the J2000 epoch. If you program these routines on a computer, you should be aware of a couple of points. First, the summation for the series should be done in reverse order to preserve as much accuracy during the operation. The equations should use the original published units. For instance, many of the equations are given in arcseconds. These units should be maintained and not converted for coding purposes. The computer program can convert the values during program execution as necessary. You can store several values (e.g. $T_{UT1}$) that are common throughout the routines, or program the entire operation as one routine. The complete process looks like the following.

**Algorithm 24: FK5 Reduction**

\[
\begin{align*}
\text{CONVTIME} (yr, mo, day, UTC, \Delta UT1, \Delta AT, x_p, y_p, z_p, UT1, TAI, TT, T_{UT1}, T_{TT})
\end{align*}
\]

Find precession and nutation angles $\Theta$, $\xi$, $\Delta \Psi$, $\epsilon$, $\theta_{AST}$

\[
\begin{align*}
[P\text{REC}] &= \text{ROT3}(-z) \text{ROT2}(\Theta) \text{ROT3}(-\xi) \\
[N\text{UT}] &= \text{ROT1}(-\epsilon) \text{ROT3}(\Delta \Psi) \text{ROT1}(\xi) \\
[ST] &= \text{ROT3}(\theta_{AST})
\end{align*}
\]

**Figure 3-28. FK5 Classical Transformation.** This figure depicts the transformation of a state vector in the body fixed (ITRF) frame to the inertial (FK5) frame. This two-way conversion is necessary for many orbit determination problems. The clear ellipses show the intermediate frames.
An example is useful to tie all the concepts together.

\textbf{Example 3-15. Performing a FK5 Reduction.}

\textbf{GIVEN:} \( \hat{\mathbf{r}}_{J2000} = 5102.5096 \mathbf{i} + 6123.01152 \mathbf{j} + 6378.1363 \mathbf{k} \text{ km} \)
\( \hat{\mathbf{v}}_{J2000} = -4.743 \mathbf{i} + 0.790 \mathbf{j}^{-536.6} + 5.533 \mathbf{k} \text{ km/s} \)

\textbf{FIND:} \( \hat{\mathbf{r}}_{ITRF}, \hat{\mathbf{v}}_{ITRF} \) on April 6, 2004, 07:51:28.386 009 UTC

Calculate the time quantities before starting the problem. Get \( \Delta \mathbf{AT} (32.0 \text{s}) \) directly from the \textit{Astronomical Almanac} (2004, K9). From the IERS's EOPCO4 final data, find \( \Delta \mathbf{UT} = -0.439 \mathbf{961.9} \text{s}, x_p = -0.140 \mathbf{682''} \) and \( y_p = 0.333 \mathbf{309''}, \) and \( \delta \Delta W = -0.052 \mathbf{195''} \) and \( \delta \Delta e = -0.003 \mathbf{875''}. \) Although general practice interpolates these values with a spline, don’t interpolate these values for this example.

\( \mathbf{UT} = \mathbf{UTC} + \Delta \mathbf{UT} = 07:51:28.386 009 - 0.439 \mathbf{969.9} = 07:51:27.946 047 \text{ UTC} \)
\( \mathbf{TAI} = \mathbf{UTC} + \Delta \mathbf{AT} = 07:51:28.386 009 + 32 \text{s} = 07:52:0.386 009 \text{ UTC} \)
\( \mathbf{TT} = \mathbf{TAI} + 32.184 \text{s} = 07:52.32.570 009 \text{ UTC} \)
\( \mathbf{DMY HMS} \Rightarrow \mathbf{JD}_{TT} = 2,453,101.828 154 745 \) (Be sure to keep all digits for accuracy.)

\( \mathbf{T}_{TT} = \frac{\mathbf{JD}_{TT} - 2,451,545.0}{36,525} = 2,453,101.828 154 745 - 2,451,545.0 = 0.042 623 631 9 \)

Now determine the rotation angles for precession from Eq. (3-57).

\( \zeta = 0.027 \mathbf{305 5''} \)
\( \Theta = 0.023 \mathbf{730 6''} \)
\( z = 0.027 \mathbf{305 9''} \)

The rotations for precession produce an intermediate vector, still in a mean-equinox, mean-equator system of date:

\( \hat{\mathbf{r}}_{mod} = 5094.029 016 \mathbf{i} + 6127.870 936 \mathbf{j} + 6380.247 888.5 \mathbf{k} \text{ km} \)
\( \hat{\mathbf{v}}_{mod} = -4.746 \mathbf{262 495} \mathbf{i} + 0.786 \mathbf{014 149} \mathbf{j} + 5.531 \mathbf{791 025} \mathbf{k} \text{ km/s} \)

Determine the angles for the Earth’s nutation (\( r = 360° \)). Notice these are the same as Example 3-14.

\( M_\odot = 314.912 287 3°, M_\odot = 91.939 376 9°, \mu M_\odot = 169.097 004 3°, \)
\[ D_\odot = 196.751 \text{ 642 8}^\circ, \Omega_\odot = 42.604 \text{ 650 0}^\circ \]

Now find the obliquity of the ecliptic [Eq. (3-53), \( \hat{\epsilon} = 23.438 \text{ 736 8}^\circ \)] and the nutation corrections for the true of date system, using the 106 coefficients from the 1980 IAU theory [Eq. (3-60)].

\[ \Delta \Psi = -0.003 \text{ 410 8}^\circ, \Delta \epsilon = 0.002 \text{ 031 6}^\circ \]

\[ \epsilon = \hat{\epsilon} + \Delta \epsilon = 23.438 \text{ 736 8} + 0.002 \text{ 031 6} = 23.440 \text{ 768 5}^\circ \]

Check that the true and mean obliquities are very close. If not, recheck your calculations! If we wished to reference the GCRF frame, we would add the corrections (\( \delta \Delta \Psi \) and \( \delta \Delta \epsilon \)) to the nutation and obliquity values.* Then, do the required rotations for nutation and find the vectors for true equinox and true equator of date:

\[ \hat{r}_{\text{tod}} = 5094.514 \text{ 781 3} \hat{i} + 6127.366 \text{ 460 9} \hat{j} + 6380.344 \text{ 532 3} \hat{k} \text{ km} \]

\[ \hat{v}_{\text{tod}} = -4.746 \text{ 088 567} \hat{i} + 0.786 \text{ 077 222} \hat{j} + 5.531 \text{ 931 288} \hat{k} \text{ km/s} \]

Determine the \( \theta_{\text{GMST}} \) using the \( JD_{\text{UT1}} \), Eq. (3-43), and Eq. (3-44):

\[ \theta_{\text{GMST}} = 312.809 \text{ 894 3}^\circ \]

Find the AST using Eq. (3-64) with the last two terms because the example occurs in 2004

\[ \theta_{\text{AST}} = \theta_{\text{GMST}} + \Delta \Psi \cos(\epsilon) = 312.806 \text{ 765 1}^\circ \]

After doing the rotation for sidereal time, you’ll get these initial \( PEF \) vectors:

\[ \hat{r}_{\text{PEF}} = -1033.475 \text{ 031 3} \hat{i} + 7901.305 \text{ 586 0} \hat{j} + 6380.344 \text{ 532 3} \hat{k} \text{ km} \]

\[ \hat{v}_{\text{PEF}} = -3.225 \text{ 632 747} \hat{i} - 2.872 \text{ 442 511} \hat{j} + 5.531 \text{ 931 288} \hat{k} \text{ km/s} \]

Finally, determine polar motion. Convert to radians and rotate to find the final Earth-centered, Earth-fixed coordinates as

\[ \hat{r}_{\text{ITRF}} = -1033.479 \text{ 383 0} \hat{i} + 7901.295 \text{ 275 8} \hat{j} + 6380.356 \text{ 595 3} \hat{k} \text{ km} \]

\[ \hat{v}_{\text{ITRF}} = -3.225 \text{ 636 520} \hat{i} - 2.872 \text{ 451 450} \hat{j} + 5.531 \text{ 924 446} \hat{k} \text{ km/s} \]

The final answer for the GCRF vector from the ITRF vector using the \( \delta \Delta \Psi \) and \( \delta \Delta \epsilon \) corrections is

\[ \hat{r}_{\text{GCRF}} = 5102.508 \text{ 958} \hat{i} + 6123.011 \text{ 401} \hat{j} + 6378.136 \text{ 928} \hat{k} \text{ km} \]

\[ \hat{v}_{\text{GCRF}} = -4.743 \text{ 220 16} \hat{i} + 0.790 \text{ 536 50} \hat{j} + 5.533 \text{ 755 73} \hat{k} \text{ km/s} \]

The length of this example and the complexity reveal the importance of knowing three things: (1) how to do the operations, (2) when you don’t need them, and (3) how to program a computer! Now examine the results of the sample problem. Table 3-6 shows the position and velocity vectors through each of the reduction rotations.

### Computational Considerations

It’s convenient to integrate the equations of motion in an inertial frame. We can integrate in a rotating coordinate system, but the process becomes a bit messy because rotating the coordinate system introduces additional terms (called Coriolis effects). One inertial frame for integrating the equations of motion is the mean equinox, mean equator of date.

---

* I’ll show the result of adding the corrections and the final GCRF vector that would produce the ITRF vector shown, but not the individual steps. The GCRF answer should be close to the FK5 (J2000) value because the offsets are small.
3.7 Transforming Celestial and Terrestrial Coordinates

This table shows the position and velocity vectors after each transformation. The terminology for mean and true values is not always followed. Mean equinox values are sometimes substituted for the true values listed.

<table>
<thead>
<tr>
<th>Position (km)</th>
<th>( \hat{r} )</th>
<th>( \hat{J} )</th>
<th>( \hat{K} )</th>
<th>Terminology</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.91*</td>
<td>5102.508958</td>
<td>6123.011401</td>
<td>6378.136928</td>
<td>( r_{GCRF} )</td>
</tr>
<tr>
<td></td>
<td>5102.509600</td>
<td>6123.011520</td>
<td>6378.136300</td>
<td>( r_{J2000} ), mean equator and mean equinox of J2000</td>
</tr>
<tr>
<td>Precession</td>
<td>5094.029016</td>
<td>6127.870936</td>
<td>6380.247888</td>
<td>( r_{mod} ), mean equator and mean equinox of date</td>
</tr>
<tr>
<td>Nutation</td>
<td>5094.514781</td>
<td>6127.366460</td>
<td>6380.344532</td>
<td>( r_{tod} ), true equator and true equinox of date</td>
</tr>
<tr>
<td>Sidereal time</td>
<td>-1033.475031</td>
<td>7901.305586</td>
<td>6380.344532</td>
<td>( \dot{r}_{PEF} )</td>
</tr>
<tr>
<td>Polar motion</td>
<td>-1033.479383</td>
<td>7901.295276</td>
<td>6380.356595</td>
<td>( \dot{r}_{ITRF} )</td>
</tr>
<tr>
<td>FK4</td>
<td>5201.586118</td>
<td>6065.401818</td>
<td>6353.101573</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Velocity (km/s)</th>
<th>( \dot{r} )</th>
<th>( \dot{J} )</th>
<th>( \dot{K} )</th>
<th>Terminology</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-4.74322016</td>
<td>0.79053650</td>
<td>5.53375573</td>
<td>( \dot{r}_{GCRF} )</td>
</tr>
<tr>
<td></td>
<td>-4.74321960</td>
<td>0.79053660</td>
<td>5.53375619</td>
<td>( \dot{r}_{J2000} )</td>
</tr>
<tr>
<td>Precession</td>
<td>-4.746262495</td>
<td>0.786014149</td>
<td>5.531791025</td>
<td>( \dot{r}_{mod} )</td>
</tr>
<tr>
<td>Nutation</td>
<td>-4.746088567</td>
<td>0.786077222</td>
<td>5.531931288</td>
<td>( \dot{r}_{tod} )</td>
</tr>
<tr>
<td>Sidereal time</td>
<td>-3.225632747</td>
<td>-2.872442511</td>
<td>5.531931288</td>
<td>( \dot{r}_{PEF} )</td>
</tr>
<tr>
<td>Polar motion</td>
<td>-3.225636520</td>
<td>-2.872451450</td>
<td>5.531924446</td>
<td>( \dot{r}_{ITRF} )</td>
</tr>
<tr>
<td>FK4</td>
<td>-4.70713917</td>
<td>0.84337306</td>
<td>5.55671671</td>
<td></td>
</tr>
</tbody>
</table>

*RMS magnitudes from the previous vectors are listed in meters. The effect of precession becomes larger as the epoch date recedes from January 1, 2000. If this example were in 2014, the deltas would be significantly larger. This shows the relative effect of each change.

J2000. However, some space centers use other “inertial” frames to integrate the equations of motion (the AF Space Command uses a true equator and mean equinox of date, others use TOD, etc.). These other frames are a legacy of earlier computers, when saving a few matrix operations (for precession and nutation) greatly accelerated system throughput. Unfortunately, these other frames are more complex because we must keep track of the epoch and frame of the integration and must transform to a new epoch and new frame when updating the reference epoch of an element set or state. The lengthy nature of the complete equations for FK5 reduction has resulted in the formation of numerous approximations. I’ll briefly discuss a four-term approximation, and the USNO NOVAS routine.
Vallado (1998) examined the behavior of the FK5 reductions. Remember that the effect of using approximate forms of the reduction matrices influences only the location of the satellite used for acceleration calculations. Figure 3-29 shows the results.

![Figure 3-29. Typical Nutation Values.](image)

This figure shows the two nutation angles and their values over time. They contain corrections from the basic and planetary series. Notice the pronounced periodic behavior.

The frequency of update influences the error.

### TABLE 3-7. Accuracy of Various Update Intervals.
Calculating daily nutation values for 20 years, this table shows the angular error resulting from updating the reduction angles at several intervals. Notice the large change from 2 to 3 weeks.

<table>
<thead>
<tr>
<th>Precession</th>
<th>1 Day</th>
<th>2 Day</th>
<th>1 Week</th>
<th>2 Week</th>
<th>3 Week</th>
<th>1 Month</th>
<th>2 Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta \Psi) (°)</td>
<td>0.000 043</td>
<td>0.000 083</td>
<td>0.000 239</td>
<td>0.000 286</td>
<td>0.000 470</td>
<td>0.000 559</td>
<td>0.001 003</td>
</tr>
<tr>
<td>(\Delta \epsilon) (°)</td>
<td>0.000 025</td>
<td>0.000 049</td>
<td>0.000 117</td>
<td>0.000 122</td>
<td>0.000 221</td>
<td>0.000 266</td>
<td>0.000 468</td>
</tr>
</tbody>
</table>

The NOVAS routine is a sophisticated method to preserve the accuracy of the full (1980) 106 term theory, but within the processing time of the 4-term approximation. The technique is originally due to Coffey and Deprit (1980) and computer source code is available on the USNO's homepage under the NOVAS routines (aa.navy.mil/AA/software/novas/novas_info.html). FORTRAN and C listings are available.

The four-term approximation is popular among users of analytical theories because it captures a portion of the nutation effect, and it requires little processing time. Examining the coefficients in the 106 term series, we can select the largest terms—be aware that list is sometimes sorted differently (See App. D). The selection of these terms is due to the fact that they are the largest terms (by at least an order of magnitude) over all other terms in the series. The AF Space Command defined a true-equator, mean equinox (TEME') system around this concept several decades ago. Its original intent was to provide an efficient coordinate system for use with their analytical theories. Despite mod-